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Initial-Value Methods for the Basic Boundary-Value Problem and Integral Equation of Radiative Transfer

HARRIET KAGIWADA and R. E. KALABA

The RAND Corporation, Santa Monica, California

ABSTRACT

We consider the problem of radiative transfer in a plane-parallel atmosphere which absorbs radiation and scatters it isotropically, there being a continuous distribution of primary internal sources of radiation. Our aim is to produce an effective numerical technique for calculating the internal intensity and the source functions. A major consideration is the computer's ability to solve initial-value problems for large systems of ordinary differential equations. We derive new differential–integral equations for the internal intensity at a fixed point and direction, and the source function at a fixed point, the new independent variable being the thickness of the atmosphere. Using standard quadrature techniques, we approximate these equations by systems of ordinary differential equations whose numerical treatment is straightforward. This work may be viewed as a new approach to the study of integral equations and linear unstable two-point boundary-value problems.

I. INTRODUCTION

Radiative transfer by an emitting, scattering, and absorbing atmosphere is one of the central topics in meteorology. It also appears in the astrophysical problems of stellar spectra and diffusion of radiation in the galaxy, and it is closely related to problems of neutron transport and thermal transport. The equations which we develop in this paper lead to an effective computational procedure for the determination of internal intensities and source functions, regardless of the nature of the geometric variation of emission rate within a slab atmosphere.

We consider anew the problem of determining the radiation field produced within a finite homogeneous slab which both absorbs radiation and scatters it isotropically, isotropic sources of radiation being located in the slab, and their strengths depending only on the altitude above the bottom.

The classical approaches to this problem [1]–[3] involve solving a two-point boundary-value problem for the transport equation (a differential-integral equation) or solving a singular integral equation for the source function. Neither of these is especially attractive from the vewpoint of modern computation. Our aim is to show how to formulate initial-value problems for both the internal intensity function and the source function. Basically, our approach is to view the internal intensity and source functions as functions of the slab thickness, position within the slab being a fixed parameter. Related previous experience indicates the computational effectiveness of the proposed methods [4]–[6].

It is important to discuss the relation between our approach and those of Carlson, Chandrasekhar–Wick, Sobolev, Ueno, and Busbridge, but this would require much additional investigation. Our ideas grew out of earlier work of Ambarzumian and Chandrasekhar, and developed within a group which includes Bellman, Wing, Ueno, and Preisendorfer. The physical situation we consider is described by Eq. (38), the integral equation for the source function. However, part of our objective is to show that we can derive the basic equations directly from the physical situation without going through the transport equation at all. That this can be done seems both philosophically and computationally interesting and not generally known.

II. Physical Situation [1], [2]

Consider a homogeneous slab of optical thickness x which absorbs radiation and reradiates it isotropically. The albedo for single scattering is denoted by λ. Located within the slab are isotropic sources of radiation. The production of energy per unit volume per unit solid angle per unit time at altitude y is $B(y)$, $0 \leq y \leq x$. A radiation field is created by the emissions and multiple scattering of this emitted radiation.

We wish to determine the intensity of radiation at the fixed altitude t, $0 \leq t \leq x$, which is propagating in a direction whose direction cosine with respect to the upward vertical is v. We denote this intensity by $I = I(t, v; x)$, which draws attention to the fact that I is to be considered a function of t, v, and x, the optical thickness. As usual, the intensity at a particular point for a particular direction is the energy per unit time per unit normal area per unit solid angle [1]–[3].

III. The Basic Equation

Let us first introduce the functions $A(x)$ and $b(t, v; x)$ in the following manner

$b(t, v; x) =$ total intensity at altitude t in a direction whose direction cosine with respect to the upward vertical is v, the slab extending from 0 to x, and due to incident radiation at the top in all downward directions of one unit of energy per unit of horizontal area per unit of solid angle per unit of time. (1)

$A(x) =$ total production of scattered radiation per unit of volume per unit of solid angle per unit of time at the top of the slab extending from 0 to x and due to the presence of all the internal sources in the slab. (2)

Then we consider the slab extending from 0 to x and add a slab of thickness Δ to the top. This results in a change in the intensity at the altitude t and in the direction with direction cosine v, due to the production of radiation in the slab added. We may write

$$I(t, v; x + \Delta) = I(t, v; x) + A(x)\Delta b(t, v; x) + o(\Delta). \qquad (3)$$

The limiting form of this equation is

$$I_x(t, v; x) = A(x)b(t, v; x), \qquad (4)$$

which is our basic relation. By also introducing the emergent intensity function

$e(v, x) = I(x, v; x)$

$=$ the intensity of the radiation emerging from the top of the slab extending from 0 to x in a direction with direction cosine v and due to all the internal sources, $0 \le v \le 1$, (5)

we can express $A(x)$ in the form

$$A(x) = B(x) + \int_0^1 e(v', x)(v')^{-1}v' \left(\frac{\lambda}{4\pi}\right) 2\pi \, dv', \qquad (6)$$

$$A(x) = B(x) + \left(\frac{\lambda}{2}\right) \int_0^1 e(v', x) \, dv'. \qquad (7)$$

IV. Additional Equations [1], [2], [6]–[7]

Presently we know that $I(t, v; x)$ satisfies the differential equation

$$I_x = b(t, v; x) \left[B(x) + \left(\frac{\lambda}{2}\right) \int_0^1 e(v', x)\, dv' \right]. \tag{8}$$

In the interest of brevity, we now simply reproduce the differential equations for b and e. These involve various other functions, h, X, Y, J, and their equations are also given:

$$b_x = \left(\frac{\lambda}{2}\right) h(t, v; x) \int_0^1 Y(u', x) \frac{du'}{u'} + v^{-1} \left[b(t, v; x) - 2 \int_0^1 J(t; x, u') \frac{du'}{u'} \right], \tag{9}$$

$$h_x = \left(\frac{\lambda}{2}\right) b(t, v; x) \int_0^1 Y(u,' x) \frac{du'}{u'}, \tag{10}$$

$$X_x = \left(\frac{\lambda}{2}\right) Y(u, x) \int_0^1 Y(u', x) \frac{du'}{u'}, \tag{11}$$

$$Y_x = -u^{-1} Y(u, x) + \left(\frac{\lambda}{2}\right) X(u, x) \int_0^1 Y(u', x) \frac{du'}{u'}, \tag{12}$$

$$J_x = -u^{-1} J(t; x, u) + \left(\frac{\lambda}{2}\right) X(u, x) \int_0^1 J(t; x, u') \frac{du'}{u'}, \tag{13}$$

$$e_x = -u^{-1} e + B(x) u^{-1} X(u, x) + \left(\frac{\lambda}{2}\right) X(u, x) u^{-1} \int_0^1 e(v', x) dv'. \tag{14}$$

The functions b and e have been defined earlier. The functions X and Y are Chandrasekhar's functions [1], [5]. The function $J(t, x, u)$ is the source function [2], [6]. (Note that t refers to optical altitude rather than depth.) The remaining function h has the same definition as b, except that the isotropic sources are at the bottom of the slab.

V. Numerical Technique

We use the method of finite ordinates to approximate the integrals occurring in Eqs. (8)–(14). In this way we obtain an approximating system of ordinary differential equations subject to known initial conditions. We introduce, for example,

$$x_i(x) = X(u_i, x), \quad i = 1, 2, \ldots, N, \tag{15}$$

where u_i is the ith root of the Nth-order shifted Legendre polynomial $P_N(1-2x)$.

The polynomials $P_0(1 - x)$, $P_1(1 - 2x)$, ... are orthogonal on the interval (0, 1). The functions $y_i(x)$, $e_i(x)$, etc., are introduced similarly. Let the corresponding Christoffel weights for Gaussian quadrature of order N be w_i, $i = 1, 2, ..., N$. For x in the interval $0 \leq x \leq t$ we consider the system of ordinary differential equations and initial conditions

$$\dot{x}_i = \left(\frac{\lambda}{2}\right) y_i \sum_{j=1}^{N} \left(\frac{y_j w_j}{u_j}\right), \tag{16}$$

$$\dot{y}_i = - u_i^{-1} y_i + \left(\frac{\lambda}{2}\right) x_i \sum_{j=1}^{N} \left(\frac{y_j w_j}{u_j}\right), \tag{17}$$

$$\dot{e}_i = - u_i^{-1} e_i + B(x) u^{-1} x_i + \left(\frac{\lambda}{2}\right) x_i u^{-1} \sum_{j=1}^{N} e_j w_j, \tag{18}$$

$$x_i(0) = 1, \tag{19}$$

$$y_i(0) = 1, \tag{20}$$

$$e_i(0) = 0, \tag{21}$$

where $i = 1, 2, ..., N$, and the dot indicates differentiation with respect to x. This system of $3N$ ordinary differential equations and initial conditions is integrated numerically for $0 \leq x \leq t$. At $x = t$, to the system of Eqs. (16)–(18) we adjoin the $N + 3$ equations

$$J_i = - u^{-1} J_i + \left(\frac{\lambda}{2}\right) x_i \sum_{j=1}^{N} \left(\frac{J_j w_j}{u_j}\right), \qquad i = 1, 2, ..., N, \tag{22}$$

and

$$\dot{l} = b\left[B(x) + \left(\frac{\lambda}{2}\right) \sum_{j=1}^{N} e_j w_j\right], \tag{23}$$

$$\dot{b} = \left(\frac{\lambda}{2}\right) h \sum_{j=1}^{N} \left(\frac{Y_j w_j}{u_j}\right) + v^{-1}\left[b - 2 \sum_{j=1}^{N} \left(\frac{J_j w_j}{u_j}\right)\right], \tag{24}$$

$$\dot{h} = \left(\frac{\lambda}{2}\right) b \sum_{j=1}^{N} \left(\frac{Y_j w_j}{u_j}\right). \tag{25}$$

Based on previous computational experience [6], and in view of the form of Eq. (24), we limit ourselves to

$$v < 0, \tag{26}$$

that is, to downward directions, to avoid numerical instability. For "initial conditions" at $x = t$, we have

$$J_i(t) = (\tfrac{1}{4}\lambda)x_i(t), \qquad i = 1, 2, \ldots, N, \tag{27}$$

$$I(t) = 0, \tag{28}$$

$$b(t) = -v^{-1}, \qquad v < 0, \tag{29}$$

$$h(t) = 0. \tag{30}$$

The resulting system of $4N + 3$ equations is integrated from $x = t$ to $x = x_0$ = desired thickness of slab. In this way we obtain the intensity at the fixed altitude t in a fixed direction with direction cosine v, $v < 0$, for a range of slab thicknesses $t \leq x \leq x_0$. For $N = 7$ and a grid size of 0.005, employing an Adams–Moulton integration scheme, the time consumed on an IBM 7044 would be in the order of a few seconds to a minute.

When $v > 0$, we make use of the relations

$$b(t, v; x) = h(x - t, -v; x), \tag{31}$$

$$h(t, v; x) = b(x - t, -v; x), \tag{32}$$

so that we may confine our calculation of b and h to nonpositive values of v. This will be explained in more detail in a paper soon to be published.

VI. Source Function J^*

We now consider the source function $J^*(t; x)$, which is defined by the relation [1], [2]

$J^*(t; x) =$ the production of scattered radiation per unit volume per unit solid angle per unit time at the altitude t in the slab extending from altitude 0 to x and due to the sources $B(y)$, $0 \leq y \leq x$. (33)

We add a slab of thickness Δ to the top and note the change that takes place in the value of J^* at the fixed altitude t, $t < x$. This leads to the equation

$$J^*(t; x + \Delta) = J^*(t; x) + A(x)\Delta \cdot 2 \int_0^1 J(t; x, u') \frac{du'}{u'} + o(\Delta), \tag{34}$$

which, in the limit as $\Delta \to 0$, becomes

$$J_x^* = A(x) \cdot 2 \int_0^1 J(t; x, u') \frac{du'}{u'}. \tag{35}$$

For numerical purposes we approximate this equation by the ordinary differential equation

$$\dot{J}^* = 2\left[B(x) + \left(\frac{\lambda}{2}\right) \sum_{j=1}^{N} e_j w_j\right] \sum_{j=1}^{N} \left(\frac{J_j w_j}{u_j}\right), \quad x \geq t. \tag{36}$$

The initial condition at $x = t$ is

$$J^*(t) = B(t) + \left(\frac{\lambda}{2}\right) \sum_{j=1}^{N} w_j e_j(t). \tag{37}$$

Ordinarily the source function $J^*(t), 0 \leq t \leq x$, is characterized as the solution of a Fredholm integral equation

$$J^*(t) = B(t) + \frac{\lambda}{2} \int_0^x J^*(s) E_1(|t - s|) ds, \tag{38}$$

where $E_1(x)$ is the exponential integral function

$$E_1(x) = \int_0^1 \exp(-x/z) \frac{dz}{z}, \quad x > 0. \tag{39}$$

In effect, we have shown how to solve this equation by integrating a system of ordinary differential equations subject to known initial conditions.

Once the source function J^* is known, the intensity function can be determined through use of the usual integral relations [1], [2].

VII. Discussion

The method presented for the numerical determination of internal and emergent radiation fields generalizes along many lines. It is clear that we may consider inhomogeneous slabs, spherical shells [8], and anisotropic scattering. In addition, we may use the ability to solve these direct problems to attack inverse problems, in which both internal and external radiation field measurements may be made and in which we wish to estimate properties of the medium [9]. We shall discuss these and related matters in subsequent papers. The reader may also wish to consult [10], where these matters are approached through the study of integral equations.

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